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BINDING INEQUALITIES FOR TREE NETWORK LOCATION PROBLEMS WITH DISTANCE CONSTRAINTS

Research Report No. 78-10

by

Richard L. Francis *
Timothy J. Lowe **
Barbaros C. Tansel *

August, 1978

*Department of Industrial and Systems Engineering
University of Florida
Gainesville, Florida 32611

**Krannert Graduate School of Management
Purdue University
West Lafayette, Indiana 47907

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In this paper, ~~we~~ considers the problem of finding locations of several new facilities in an imbedded tree network with respect to existing facilities at known locations so as to satisfy distance constraints, which impose upper bounds on distances between pairs of facilities. It is known that the existence of a feasible solution to the distance constraints is related to shortest paths through an auxiliary network, which has as arc lengths the upper bounds on pairwise facilities distances. This relationship takes the form of → next page

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necessary and sufficient conditions, termed the separation conditions. ~~We relate~~ "tight" separation conditions are related to the solution of multifacility minimax location problems and efficient solutions to multiobjective multifacility location problems. In addition, we provide a proof that the SLP algorithm of [8] is in fact an algorithm of lowest order of computation for determining whether or not the distance constraints are consistent.

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Abstract

In this paper, we consider the problem of finding locations of several new facilities in an imbedded tree network with respect to existing facilities at known locations so as to satisfy distance constraints, which impose upper bounds on distances between pairs of facilities. It is known that the existence of a feasible solution to the distance constraints is related to shortest paths through an auxiliary network, which has as arc lengths the upper bounds on pairwise facility distances. This relationship takes the form of necessary and sufficient conditions, termed the separation conditions. We relate "tight" separation conditions to the solution of multifacility minimax location problems and efficient solutions to multiobjective multifacility location problems. In addition, we provide a proof that the SLP algorithm of [8] is in fact an algorithm of lowest order of computation for determining whether or not the distance constraints are consistent.

1. Introduction

Network location problems involve locating new facilities on a network, such as a transport network, with respect to existing facilities lying on the network at known locations. For such problems distances between facilities are of interest: these distances are actual "shortest route distances" as determined from the network. Associated with many network location problems is a collection of distance constraints. These distance constraints specify known upper bounds on distances between pairs of facilities (pairs of new facilities, and pairs of new and existing facilities) so that the facilities will not be too far apart from one another.

Distance constraints appear to be a recurring feature in network location problems. In the problem of deciding where to locate a fire station [18], it may be relevant to place restrictions on the location via distance constraints so that the response time to the site of any potential fire may not be too large. The location of booster pumps in a pipeline system might well involve distance constraints in order to locate the pumps so that the pressure in the pipeline does not fall below a certain level. Also, certain minimax location problems on networks have an equivalent formulation in terms of distance constraints [8], since typical objective functions in such problems involve distances between facilities, permitting equivalent formulations with distances as constraints.

Our work extends earlier results [8] obtained for distance constraints for the case where the underlying network is a tree, that is, it is connected, undirected, has positive arc lengths and has no cycles. Such networks tend to occur when only a "sparse" network is justifiable because of limited need for the network or, alternatively, when it is very expensive to provide redundant links (as in some road networks). In addition, some pipeline, and associated electrical control networks, are tree networks [3].

We now give a brief overview of our main results; the necessary notation and definitions appear in Section 2, including the "distance constraints." Also we state the "separation conditions", necessary and sufficient conditions for a feasible solution to exist to the distance constraints. It is shown in [8] that in order for the separation conditions to be both necessary and sufficient for the existence of a feasible solution to the distance constraints the network must be a tree. Thus in this paper we restrict ourselves to the case where the network is a tree. In Section 3 we establish preliminary results and study "tight paths", paths which involve an auxilary network defined by the distance constraints. The properties of Section 3 provide the basis for our subsequent results.

In Section 4 we give necessary and sufficient conditions for a feasible solution to be an optimal feasible solution to a minimax location problem, and present an example. Minimax location problems occur when, for example, new facilities are to be placed so as to minimize the maximum of travel times between given pairs of new facilities, and given pairs of new and existing facilities. Thus, the new facilities are to be placed so as to provide quick "service" to existing facilities, while at the same time being able to provide service to one another. We list as references a number of papers which consider minimax location problems on networks (See references [4], [5], [9], [10], [14], [15]).

In Section 5 we study a multiobjective location problem for which it is desired to locate certain pairs of new facilities and certain pairs of new and existing facilities as close together as possible. Efficient location vectors to such problems are such that in order for any new facility to be closer to some facility than it already is, it must in turn be placed farther from some other facility. This particular multiobjective location problem is similar in spirit to a problem in the plane using rectilinear distances studied by Wendell, Hurter and Lowe [19], and Chalmet and Francis [1]. Other examples of multiobjective location problems on networks can be found in [11], [12], [13] and [16].

Also in Section 5 we employ the idea of efficient location vectors to demonstrate that any algorithm which checks the distance constraints for consistency (i.e., whether or not the distance constraints have a feasible solution) is, for the worst case, at least of order $m^2 + mn$, where m and n are the number of new and existing facilities respectively. We further point out that the "SLP" algorithm of [8] is of order $m^2 + mn$, so that, in a well defined sense, the algorithm is in fact a "best" algorithm.

Throughout Sections 4 and 5 we rely heavily on properties of tight paths, which provide the underlying unifying principles in our work.

Section 2 Notation, Definitions, and Statements of Results.

We suppose we are given T, an imbedded undirected tree having positive arc lengths, such that the distance $d(x, y)$ between points $x, y \in T$ is the well-defined shortest path distance and is a metric, as discussed in [5]. We are given existing facility locations $a_1, a_2, \ldots, a_n$ in T. We wish to find new facility locations $x_1, \ldots, x_m$ in T, where x_i is the location of new facility i, $i = 1, \ldots, m$.

As in [6], given points $x, y \in T$, we define the line $L(x, y)$ as the union of all points in the shortest path between x and y; equivalently, $L(x,y) = \{z \in T: \; d(x,z) + d(z,y) = d(x,y)\}$. In addition, given a collection of points $P = \{P_1, \ldots, P_k\}$ contained in T, we let $H(P)$ denote the convex hull [6] of P, the smallest (imbedded) subtree of T containing P. Corresponding to the n existing facilities and m new facilities, we are given a set of Distance Constraints, which for convenience we refer to as DC, as follows:

$$\begin{aligned} d(x_j, x_k) &\leqq b_{jk}, \quad (j, k) \in I_B, \\ d(x_i, a_j) &\leqq c_{ij}, \quad (i, j) \in I_C, \end{aligned} \tag{2.1}$$

where the b_{jk} and c_{ij} are known positive constants. We remark that it need not be the case that the set I_B includes all possible pairs of new facility indices, nor I_C includes all possible pairs of new and existing facility indices.

Corresponding to DC, we define Network BC (NBC) as the undirected network having nodes $E_1, \ldots, E_n, N_1, \ldots, N_m$; for every $(j, k) \in I_B$, there is an arc (N_j, N_k) of length b_{jk} between nodes N_j and N_k; for every $(i, j) \in I_C$, there is an arc (N_i, E_j) of length c_{ij} between nodes N_i and E_j. We further assume that the sets I_B and I_C are such that NBC is connected, as otherwise DC decomposes into independent sets of constraints which may be analyzed separately.

Given a node-path between any two nodes f_p and f_q in NBC, we denote the path by $P(f_p, f_q)$ and denote the length of the path by $LP(f_p, f_q)$. We define $\mathcal{L}(f_p, f_q)$ to be the <u>length of any shortest path</u> in NBC between nodes f_p and f_q. Subsequently, unless we specify otherwise, it should be understood that any path we refer to is a simple path between some two nodes E_p and E_q.

The following result is established in [8]:

<u>Theorem 2.1</u> DC is consistent if and only if

$$d(a_p, a_q) \leqq \mathcal{L}(E_p, E_q), \; 1 \leqq p < q \leqq n. \tag{2.2}$$

The inequalities (2.2) are termed the <u>Separation Conditions</u> [8], since each term on the right specifies an upper bound on how separate two existing facility locations can be. Except when stated otherwise, we assume throughout this paper that the separation conditions hold, and thus (equivalently) DC is consistent.

We call a path $P(E_p, E_q)$ between E_p and E_q in NBC a <u>tight path</u> if $LP(E_p, E_q) = d(a_p, a_q)$. We note that since we assume DC is consistent, it necessarily follows if $P(E_p, E_q)$ is a tight path, that $LP(E_p, E_q) = \mathcal{L}(E_p, E_q)$. Any path $P(E_p, E_q)$ for which $LP(E_p, E_q) > d(a_p, a_q)$ is called a <u>slack path</u>, or <u>loose path</u>.

We say that new facility i is <u>in a tight path</u> if there exists at least one tight path containing N_i. Every path containing N_i is slack if there is no tight path which contains N_i.

The motivation for the above terminology is due to a string network representation of NBC. This string network is also useful for obtaining problem insights. When knots representing nodes E_p and E_q are pulled as for apart as possible, the distance between the two knots is $\mathcal{L}(E_p, E_q)$. If then the string network is placed upon the tree T, i.e. the strings only lie on arcs of T, a path is tight when it is necessary to pull the string network tight in order to place the knots representing E_p and E_q on a_p and a_q respectively, while a path is slack if the string path must literally be slack when the two knots are placed to coincide with a_p and a_q.

A priori, one might think that the occurrence of a tight path would be rare. However, we shall see in Sections 4 and 5 that tight paths occur in a quite natural way when the separation conditions are used in the analysis of minimax and of efficient location problems. Further, the notion of tight paths permits the specification of necessary and sufficient conditions for DC to have a unique solution.

Section 3 Preliminary Analysis and Properties of Tight Paths

In this section, we establish some preliminary lemmas, and then present some properties of tight paths. We will find the following three lemmas of use in our analysis. The proof of Lemma 3.1 can be found in [7].

Lemma 3.1 Given a, b, x ε T with $d(a, b) = \alpha + \beta$, $d(x, a) = \alpha$ and $d(x, b) = \beta$, if $d(y, a) \leqq \alpha$ and $d(y, b) \leqq \beta$, then $x = y$.

Lemma 3.2 Given a, b ε T, $d(a, b) = \alpha + \beta$, the inequalities $d(x, a) \leqq \alpha$, and $d(x, b) \leqq \beta$, are consistent if and only if they have a unique solution and the inequalities hold as equalities.

Proof: By hypothesis and the triangle inequality, there exists x ε T where $d(a, b) \leqq d(a, x) + d(x, b) \leqq \alpha + \beta = d(a, b)$, which with the hypotheses, clearly implies $d(a, x) = \alpha$ and $d(x, b) = \beta$. Lemma 3.1 implies uniqueness. The proof of the converse is trivial, and we omit it.

Lemma 3.3 Given x_0, x_{r+1} ε T and $d(x_0, x_{r+1}) = b_1 + b_2 + \ldots + b_{r+1}$, the inequalities

$$d(x_i,\ x_{i+1}) \leqq b_{i+1}\ ,\ i = 0,\ \ldots,\ r\ , \tag{3.1}$$

are consistent if and only if they have a unique solution and

$$d(x_i,\ x_{i+1}) = b_{i+1}\ ,\ i = 0,\ \ldots,\ r. \tag{3.2}$$

Further,

$$x_i \ \varepsilon\ L(x_{i-1},\ x_{i+1})\ ,\ i = 1,\ \ldots,\ r. \tag{3.3}$$

Proof: Given (3.1) is consistent, and using the triangle inequality and (3.1) for any j, j = 1, ..., r,

$$d(x_0,\ x_j) \leqq d(x_0,\ x_1) + \ldots + d(x_{j-1},\ x_j) \leqq b_1 + \ldots + b_j \equiv \alpha_j\ ,$$

and

$$d(x_j,\ x_{r+1}) \leqq d(x_j,\ x_{j+1}) + \ldots + d(x_r,\ x_{r+1}) \leqq b_{j+1} + \ldots + b_{r+1} \equiv \beta_j.$$

Further, it is clear that $d(x_0,\ x_{r+1}) = \alpha_j + \beta_j$.

Define $y_0 = x_0$, $y_{r+1} = x_{r+1}$, and suppose

$$d(y_i,\ y_{i+1}) \leqq b_{i+1}\ ,\ i = 1,\ \ldots,\ r. \tag{3.4}$$

Using (3.4) and the triangle inequality, we conclude for any j, j = 1, ..., r, that $d(y_0,\ y_j) \leqq \alpha_j$, and $d(y_j,\ y_{r+1}) \leqq \beta_j$.

Since $x_0 = y_0$ and $x_{r+1} = y_{r+1}$, it follows that $d(x_0,\ y_j) \leqq \alpha_j$ and $d(y_j,\ x_{r+1}) \leqq \beta_j$. Lemma 3.2 now implies $y_j = x_j$, j = 1, ...,r. Repeated use of the triangle inequality gives $b_1 + b_2 + \ldots + b_{r+1} = d(x_0,\ x_{n+1})$ $\leqq d(x_0,\ x_1) + \ldots + d(x_r,\ x_{r+1}) \leqq b_1 + \ldots + b_{r+1}$, from which it follows, using also (3.1), that (3.2) is true. We omit the (trivial) proof of the converse.

To show (3.3), for any i, note, in either case, that

$$\begin{aligned} b_1 + b_2 + \ldots + b_{r+1} &= d(x_0,\ x_{r+1}) \\ &\leqq d(x_0,\ x_{i-1}) + d(x_{i-1},\ x_{i+1}) + d(x_{i+1},\ x_{r+1}). \end{aligned} \tag{3.5}$$

But then

$$d(x_0, x_{i-1}) \leqq b_1 + \ldots + b_{i-1},$$

$$d(x_{i-1}, x_{i+1}) \leqq b_i + b_{i+1}, \tag{3.6}$$

$$d(x_{i+1}, x_{r+1}) \leqq b_{i+2} + \ldots + b_{r+1}.$$

Using (3.5) and (3.6), it follows that $d(x_{i-1}, x_{i+1}) = b_i + b_{i+1}$, which along with $d(x_{i-1}, x_i) = b_i$ and $d(x_i, x_{i+1}) = b_{i+1}$, implies (3.3).

We now relate unique locations to tight paths. By definition, new facility i is uniquely located if it has the same location in every feasible solution to DC. Since we later refer to a collection of facilities, which contains possibly both existing and new facilities, being uniquely located, we note that existing facilities are uniquely located by definition.

We now state Property 3.1, the proof of which follows from Lemmas 3.4 and 3.5 below.

Property 3.1: New facility k is uniquely located if and only if N_k lies on at least one tight path $P(E_p, E_q)$.

As an immediate consequence of Property 3.1 we have

Corollary 3.1: DC has a unique solution if and only if N_k lies on at least one tight path in NBC for $k = 1, \ldots, m$.

We now present Lemmas 3.4 and 3.5.

Lemma 3.4: If N_k lies on at least one tight path $P(E_p, E_q)$ in NBC, then new facility k is uniquely located.

Proof: Index the nodes in the tight path $P(E_p, E_q)$ as $(f_0, f_1, f_2, \ldots, f_r, f_{r+1})$, where $f_0 = E_p$, $f_{r+1} = E_q$ and each f_j, $2 \leqq j \leqq r$, is an N or E node. (We note that since no two E nodes are adjacent in NBC, no two adjacent nodes in $(f_0, f_1, \ldots, f_{r+1})$ are E nodes.)

Letting b_j be the length of arc (f_{j-1}, f_j), $j = 1, \ldots, r + 1$; since $P(E_p, E_q)$ is a tight path we have

$$d(a_p, a_q) = b_1 + b_2 + \ldots + b_{r+1}. \tag{3.7}$$

Since DC is consistent, there exists a location vector $X = (x_1, x_2, \ldots, x_m)$ which satisfies the distance constraints. Let j be arbitrary, $0 \leqq j \leqq r$.

If $f_j = N_{(j)}$ and $f_{j+1} = E_{(j+1)}$, then $((j), (j+1)) \varepsilon I_C$ and

$$d(x_{(j)}, a_{(j+1)}) \leqq c_{(j),(j+1)} \equiv b_{j+1}. \tag{3.8}$$

If $f_j = E_{(j)}$ and $f_{j+1} = N_{(j+1)}$, then $((j+1), (j)) \varepsilon I_C$ and

$$d(a_{(j)}, x_{(j+1)}) = d(x_{(j+1)}, a_{(j)}) \leqq c_{(j+1),(j)} \equiv b_{j+1}. \tag{3.9}$$

If $f_j = N_{(j)}$ and $f_{j+1} = N_{(j+1)}$, then $((j), (j+1)) \varepsilon I_B$ and

$$d(x_{(j)}, x_{(j+1)}) \leqq b_{(j),(j+1)} \equiv b_{j+1}. \tag{3.10}$$

But then, from (3.7), (3.8), (3.9), and (3.10), the hypotheses of Lemma 3.3 are satisfied and so new facility k is uniquely located. The conclusion now follows.

We next establish the converse of Lemma 3.4.

Lemma 3.5: If new facility k is uniquely located then N_k lies on at least one tight path $P(E_p, E_q)$ in NBC.

Proof: We shall show equivalently that if every path containing N_k is slack, then there exist solutions to DC with distinct locations of new facility k, where, without loss of generality, we let $k = 1$. Assume all paths through N_1 are slack. It is then the case that any such path with "least" slack has positive slack. That is, there are existing facility nodes, taken to be E_1 and E_2 without loss of generality, and a path $\bar{P}_{12} \equiv P(E_1, E_2)$ which contains N_1, such that, with $d_{12} \equiv d(a_1, a_2)$, $\beta \equiv L\bar{P}_{12} - d_{12} > 0$ and, for any E_s and E_t and any path $\tilde{P}_{st} \equiv \tilde{P}(E_s, E_t)$ containing N_1, with $d_{st} \equiv d(a_s, a_t)$, it is true that

$$L\hat{P}_{st} \geq \beta + d_{st} \, . \tag{3.11}$$

Since $\bar{P}_{12}$ has least slack, it is a shortest path through N_1 joining E_1 and E_2 and thus $L\bar{P}_{12} = \mathcal{S}(E_1, N_1) + \mathcal{S}(E_2, N_1)$. As a notational simplification, we let $\ell_i = \mathcal{S}(E_i, N_1)$, $i = 1, 2$.

Let DC_i , $i = 1, 2$, be the set of distance constraints consisting of DC in addition to the constraint $d(x_1, a_i) \leqq \gamma_i$. We note that any feasible solution to DC_i, $i = 1, 2$, is a feasible solution to DC. In what follows we show that by proper choice of γ_i, DC_i is consistent, $i = 1, 2$, and that if X^1 is feasible to DC_1 and X^2 is feasible to DC_2, then x_1^1 and x_1^2 are distinct, where x_1^i is the first component of X^i, $i = 1, 2$.

Let NBC_i be the network associated with DC_i. We note that NBC_i differs from NBC only in that it contains the additional arc between nodes E_i and N_1 of length γ_i. Label this additional arc α_i.

Let E_s and E_t be any two existing facility nodes in NBC_i and let $\bar{P}_{st}$ be a shortest path in NBC_i of length $L\bar{P}_{st}$, joining E_s and E_t. We shall show that if γ_i is positive and if

$$\gamma_i \geqq \ell_i - \beta \quad , \tag{3.12}$$

then,

$$L\bar{P}_{st} \geqq d_{st}, \tag{3.13}$$

and thus, due to Theorem 2.1, DC_i is consistent. We shall show that (3.13) holds when $i = 1$. Due to symmetry, the case when $i = 2$ will follow.

If $\bar{P}_{st}$ contains only arcs which are arcs of NBC, i.e., $\bar{P}_{st}$ does not contain α_1, then (3.13) is certainly true since the separation conditions hold for DC. Thus suppose $\bar{P}_{st}$ contains α_1 (and hence contains N_1). $\bar{P}_{st}$ can be decomposed as $\bar{P}_{st} = (P_1, \alpha_1, P_2)$, where P_1 and P_2 are (possibly empty) paths joining E_s and E_1, and N_1 and E_t, respectively. Since $\bar{P}_{st}$ is a shortest path, neither P_1 nor P_2 contains α_1, and thus P_1 and P_2 (when non-empty) contain only arcs which are arcs of NBC. The length of $\bar{P}_{st}$ is $L\bar{P}_{st} = LP_1 + \gamma_1 + LP_2$, and using (3.12) we obtain

$$L\bar{P}_{st} \geqq LP_1 + \ell_i - \beta + LP_2, \text{ or}$$

$$L\bar{P}_{st} + \beta \geqq LP_1 + \ell_1 + LP_2. \tag{3.14}$$

Noting that the right hand side of (3.14) is the length, $L\tilde{P}_{st}$, of a path, $\tilde{P}_{st}$, in NBC between E_s and E_t which contains N_1, (3.13) follows from (3.11) and (3.14). (We remark, to motivate the requirement (3.12), that if, say, $\gamma_1 < \ell_1 - \beta$, then $\gamma_1 + \ell_2 < \ell_1 + \ell_2 - \beta = d_{12}$: but $\gamma_1 + \ell_2$ is the length of a path in NBC_1, not entirely contained in NBC, and so Theorem 2.1 would imply DC_1 is inconsistent.)

Define the positive quantity $\varepsilon = \min\{\ell_1, \ell_2, d_{12}\}$, and let γ_i be the positive quantity, which we note satisfies (3.12), defined as $\gamma_i = \max\{^{\varepsilon}/_3, \ell_i - \beta\}$, i = 1, 2. Since DC_i is consistent, let X^i be a feasible solution to DC_i, and let x_1^i be the first component of X^i, i =1, 2.

For x_1^1 we have $d(a_1, x_1^1) \leqq \gamma_1 = \max\{^{\varepsilon}/_3, \ell_1 - \beta\}$, and so the triangle inequality gives $d(a_2, x_1^1) \geqq d_{12} - d(a_1, x_1^1) \geqq d_{12} - \max\{^{\varepsilon}/_3, \ell_1 - \beta\} = \min\{d_{12} - {}^{\varepsilon}/_3, d_{12} - \ell_1 + \beta\}$. Noting that $d_{12} - \ell_1 + \beta = \ell_2$ we obtain

$$d(a_2, x_1^1) \geqq \min\{d_{12} - {}^{\varepsilon}/_3, \ell_2\}. \tag{3.15}$$

For x_1^2 we have

$$d(a_2, x_1^2) \leqq \gamma_2 = \max\{^{\varepsilon}/_3, \ell_2 - \beta\}. \tag{3.16}$$

Using the definition of ε, it follows that the right hand side of (3.15) is strictly larger than the right hand side of (3.16) and thus x_1^1 and x_1^2 are distinct. Noting that X^i, i = 1, 2, is feasible to DC completes the proof.

We can now draw some additional conclusions about tight paths.

Property 3.2: If $P(E_p, E_q)$ is a tight path, then the nodes representing facilities in the path occur with the same ordering and spacing in the path as do the locations representing the facilities in $L(a_p, a_q)$. Further, every facility represented by a node in $P(E_p, E_q)$ is uniquely located.

Proof: The proof is an immediate consequence of Lemmas 3.3 and 3.4.

We emphasize the fact that the knowledge of a tight path immediately identifies the location of every new facility represented by a node on the tight path. For example, if $P(E_1, E_3) = (E_1, N_1, N_2, E_2, N_3, E_3)$ is a tight path, with $c_{11} = 2$, $b_{12} = 3$, $c_{22} = 1$, $c_{32} = 4$, $c_{33} = 2$, then $d(a_1, a_3) = 12$, and so new facility 1 would be located in $L(a_1, a_3)$ such that $d(x_1, a_1) = 2$, new facility 2 would be located in $L(a_1, a_3)$ such that $d(x_2, a_1) = 5$, and new facility 3 would be located in $L(a_1, a_3)$ such that $d(x_3, a_1) = 10$.

We now consider the problem of determining when an arc lies on a tight path. As an arc lies on a tight path if and only if it is not the case that all paths containing the arc are slack, we consider the equivalent problem of determining when an arc lies only on slack paths.

Property 3.3: Let DC be consistent. Let (f_i, f_j) be any arc in NBC, of length e_{ij}, whose length is reduced by some positive amount ε. Let $DC_\varepsilon(NBC_\varepsilon)$ be the distance constraints (network) obtained from DC(NBC) by replacing e_{ij} by $e_{ij} - \varepsilon$.

(a) Every path containing (f_i, f_j) in NBC is slack if and only if ε can be chosen (with $\varepsilon > 0$) so that DC_ε is consistent.

(b) Whenever every path containing (f_i, f_j) is slack, ε can be chosen (with $\varepsilon > 0$) so that DC_ε is consistent and at least one of the following is true:

(i) at least one path in NBC_ε containing (f_i, f_j) is tight;

(ii) the length of (f_i, f_j) in NBC_ε can be reduced to zero.

<u>Proof</u>: Assume ε can be chosen (with $\varepsilon > 0$) so that DC_ε is consistent. Theorem 2.1 then implies the separation conditions hold for DC_ε. If at least one path, say $P(E_p, E_q)$, in NBC which contains (f_i, f_j) is tight, then $d(a_p, a_q) = LP(E_p, E_q)$, so that reducing e_{ij} by ε would give $LP_\varepsilon(E_p, E_q) \equiv LP(E_p, E_q) - \varepsilon < d(a_p, a_q)$. But $LP_\varepsilon(E_p, E_q)$ is the length of $P(E_p, E_q)$ in NBC_ε, and hence a separation condition for DC_ε is violated for any ε, $\varepsilon > 0$, giving a contradiction. Hence every path containing (f_i, f_j) in NBC is slack.

Conversely, assume every path containing (f_i, f_j) in NBC is slack, so that

$$LP(E_p, E_q) - d(a_p, a_q) > 0 \tag{3.17}$$

for every path $P(E_p, E_q)$ containing (f_i, f_j), and define ε' to be the minimum of the left side of (3.17) over all such paths, giving $\varepsilon' > 0$. Let $\varepsilon = \min(\varepsilon', e_{ij}) > 0$. Since any path $P(E_p, E_q)$ containing (f_i, f_j) has its length reduced to $LP_\varepsilon(E_p, E_q) \equiv LP(E_p, E_q) - \varepsilon$, the inequality

$$d(a_p, a_q) \leqq LP_\varepsilon(E_p, E_q) \tag{3.18}$$

is equivalent to

$$\varepsilon \leq LP(E_p, E_q) - d(a_p, a_q). \tag{3.19}$$

The definition of ε' and $\varepsilon \leq \varepsilon'$ imply (3.19) is true, and so (3.18) is true. Further, if $\varepsilon = \varepsilon'$, then (3.19) holds as an equality for some $P(E_p, E_q)$, implying its length in NBC_ε is zero, giving (b) - (i). The case $\varepsilon = e_{ij}$ gives (b) - (ii). (Parenthetically, we note that the largest ε can be and still have DC_ε consistent is thus clearly $\max(\varepsilon', e_{ij})$.) As any path $P(E_p, E_q)$ in NBC_ε not containing (f_i, f_j) has its length unchanged from that in NBC, DC consistent and Theorem 2.1 implies the path satisfies its separation condition for DC_ε.

Hence, with (3.18), the separation conditions for DC_ε are satisfied, so Theorem 2.1 implies DC_ε is consistent, completing the proof.

Section 4: A Minimax Location Problem Given m new facilities at locations $x_1, \ldots, x_m$ to be determined, existing facilities at known locations $a_1, \ldots, a_n$, and index sets I_B, I_C as before, the minimax location problem may be stated as follows:

(PMM) minimize z

subject to

$$v_{jk}\, d(x_j, x_k) \le z \quad , (j, k) \in I_B$$

$$w_{ij}\, d(x_i, a_j) \le z \quad , (i, j) \in I_C.$$

Here the v_{jk} and w_{ij} are given positive weights, and the problem becomes one of locating the new facilities so as to minimize the maximum of the weighted distances. To define the network BC of interest for this problem, define $b_{jk} = 1/v_{jk}$ for $(j, k) \in I_B$, and define $c_{ij} = 1/w_{ij}$ for $(i, j) \in I_C$.

The following result for this problem is proven in [8]:

Lemma 4.1: The minimum objective function value for (PMM) is given by

$$z^* = \max[d(a_p, a_q)/\mathcal{L}(E_p, E_q) : 1 \le p < q \le m].$$

As is pointed out in [8], one way to solve the minimax problem is to use Lemma 4.1, set $z = z^*$ in the constraints of the problem, and use the SLP algorithm to construct a feasible solution to the constraints; any such feasible solution is optimal.

Our interest here is to explore relationships between Lemma 4.1 and tight paths. To this end, it will be convenient to denote by NBC(z) the network obtained from NBC by multiplying every arc length in NBC by z. We now state

Property 4.1: Let (X, z) be a feasible solution to (PMM).

(a) (X, z) is an optimum feasible solution to (PMM) if and only if at least one path in NBC(z) is tight, that is, for some $P(E_j, E_k)$.

$$d(a_j, a_k) = z\ LP(E_j, E_k).$$

(b) For any such tight path, the facilities whose nodes lie on the path are uniquely located, and their locations have the same ordering and spacing in T as their nodes have in the corresponding path in NBC.

Proof: (a) Suppose $P(E_j, E_k)$ is a tight path, so that

$$d(a_j, a_k) \geq z\ LP(E_j, E_k). \tag{4.1}$$

The definition of $\mathcal{L}(E_j, E_k)$ and $z > 0$ give

$$z\ LP(E_j, E_k) \geq z\ \mathcal{L}(E_j, E_k), \tag{4.2}$$

while z* the minimum objective function value (by Lemma 4.1) gives

$$z\ \mathcal{L}(E_j, E_k) \geq z^* \mathcal{L}(E_j, E_k). \tag{4.3}$$

Hence Lemma 4.1 gives

$$\begin{aligned} z^* \mathcal{L}(E_j, E_k) &\geq [d(a_j, a_k)/\mathcal{L}(E_j, E_k)]\mathcal{L}(E_j, E_k) \\ &= d(a_j, a_k). \end{aligned} \tag{4.4}$$

Thus we conclude the inequalities (4.1) through (4.4) all hold as equalities, implying $z = z^*$, and hence (X, z) is an optimum feasible solution.

Conversely, suppose (X, z) is an optimum feasible solution. Lemma 4.1 then gives, for some $P(E_j, E_k)$,

$$\begin{aligned} z = z^* &= \max[d(a_p, a_q)/\mathcal{L}(E_p, E_q) : 1 \leq p < q \leq n] \\ &= d(a_j, a_k)/\mathcal{L}(E_j, E_k) \\ &= d(a_j, a_k)/LP(E_j, E_k), \end{aligned}$$

implying $P(E_j, E_k)$ is a tight path in NBC(z).

(b) The proof is immediate from (a) and Properties 3.1 and 3.2.

We now present an example. Suppose a tree T is given as shown in Fig. 1, with three new facilities and three existing facilities (at a_1, a_2, and a_3). $I_B = \{(1, 2), (1, 3), (2, 3)\}$, $I_C = \{(1, 1), (2, 2), (3, 3)\}$, $v_{12} = 1$, $v_{13} = 10$, $v_{23} = 1$, $w_{11} = 10$, $w_{22} = 2$, and $w_{33} = 5$. Fig. 2 illustrates NBC with $b_{jk} = 1/v_{jk}$ and $c_{ij} = 1/w_{ij}$. It is readily verified that $\mathcal{L}(E_1, E_2) = 1.6$, $\mathcal{L}(E_1, E_3) = .4$, $\mathcal{L}(E_2, E_3) = 1.7$, $d(a_1, a_2) = 5$, $d(a_1, a_3) = 6$, and $d(a_2, a_3) = 7$. Thus Lemma 4.1 gives $z^* = \max[5/1.6, 6/.4, 7/1.7] = 15 = d(a_1, a_3)/\mathcal{L}(E_1, E_3)$. Hence, in NBC($z^*$), Property 4.1 implies the path (E_1, N_1, N_3, E_3) is tight, with total length $(.1)(15) + (.1)(15) + (.2)(15) = 6 = d(a_1, a_3)$, and so new facility 1 and new facility 3 are uniquely located at x_1^*, x_3^* respectively in $L(a_1, a_3)$, where x_1^* is the point in $L(a_1, a_3)$ such that $d(a_1, x_1^*) = 1.5$, and x_3^* is the point in $L(a_1, a_3)$ such that $d(a_1, x_3^*) = 3$. It is easy to verify that N_2 lies only on loose paths in NBC(z^*), and hence new facility 2 is not uniquely located. In fact, the inequalities $2d(x_2^*, a_2) \leq 15$, $1d(x_2^*, x_1^*) \leq 15$, and $1d(x_2^*, x_3^*) \leq 15$ permit x_2^* to be any point in the tree T. This example illustrates the fact that the knowledge of tight paths in NBC(z) permits one to determine immediately some new facility locations for (PMM), as well as identify those facilities which have "critical" locations, in the sense that a change in their locations would cause z^* to change as well.

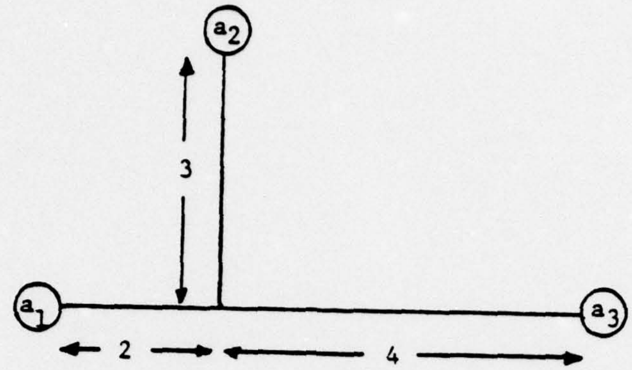

Figure 1. Tree for Example Minimax Problem

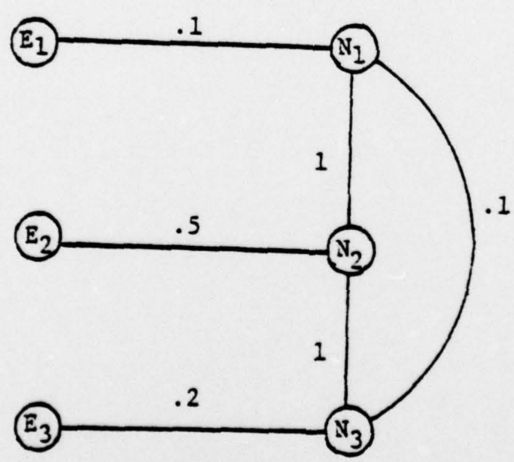

Figure 2. NBC for Example Minimax Problem

Section 5 Efficient Location Vectors

As before we suppose existing facility locations $a_1, \ldots, a_n$ to be given, and that new facility locations $x_1, \ldots, x_m$ are of interest. Given any vector of new facility locations $Z = (z_1, \ldots, z_m)$, we denote by $D(Z)$ the vector having the entries $d(z_j, z_k)$ for $(j, k) \varepsilon I_B$, and $d(z_i, a_j)$ for $(i, j) \varepsilon I_C$, where I_B and I_C are given pairs of facility indices. A location vector Z is dominated if there exists a location vector Y such that $D(Y) \leq D(Z)$ and $D(Y) \neq D(Z)$. A location vector Z which is not dominated is called efficient. Thus Z is efficient if and only if $D(Y) \leq D(Z)$ implies $D(Y) = D(Z)$. Hence, whenever Z is efficient, if a location vector X is given such that some entries in D(X) are (strictly) less than the corresponding entries in D(Z), then it must be true that at least one entry in D(X) is (strictly) greater than the corresponding entry in D(Z). One can consider the problem of finding efficient location vectors as a multiobjective optimization problem, with one optimizer for every entry in D(Z); the optimizers can agree that dominated location vectors are not of interest, thus leaving the efficient location vectors to consider. Even if there is only a single optimizer, and his objective function is strictly increasing in each entry of D(Z), then every location vector which minimizes his objective function is clearly efficient, so that knowledge of efficient location vectors may facilitate a sensitivity analysis.

As an example of such an objective function, suppose positive "weights" v_{jk} and w_{ij} are given, and define f(Z) by f(Z) =
$\Sigma\{v_{jk}\, d(z_j, z_k) : (j, k) \varepsilon I_B\} + \Sigma\{w_{ij}\, d(z_i, a_j) : (i, j) \varepsilon I_C\}$.
Picard and Ratliff [17] have recently presented a means of finding new facility locations in a tree to minimize f(Z), and discuss related

literature, which is extensive. Whenever a single optimizer has a continuous objective function (such as $f(Z)$), then the compactness of T and the extreme value theorem asserts there will always exist a location vector optimizing his objective function, in turn implying that efficient location vectors always exist for nonpathological problems. A further discussion of multiobjective location problems on networks can be found in [11], [12], [13], and [16].

Given a location vector Z, we let $U = D(Z)$ and define the distance constraints of interest by $D(X) \leq U$, where the entries in U define the b_{jk} and c_{ij} by $b_{jk} = d(z_j, z_k)$ for $(j, k) \varepsilon I_B$, and $c_{ij} = d(z_i, a_j)$ for $(i, j) \varepsilon I_C$. We use the b_{jk} and c_{ij} to define NBC in the customary manner. As before, we may assume NBC is connected, for otherwise the problem of finding efficient location vectors decomposes into independent subproblems. Further, we note that DC is always consistent, as Z is certainly feasible to DC, and hence, by Theorem 2.1, the separation conditions are always satisfied. For convenience, for any location vector Z, we denote by $A_i(Z)$ the collection of locations of uniquely located facilities whose nodes are adjacent to N_i in NBC. We denote by $H[A_i(Z)]$ the convex hull of $A_i(Z)$, the imbedding of the smallest subtree of T spanning all the elements of $A_i(Z)$.

With the above definitions we can present a family of equivalent conditions for a location vector Z to be efficient.

Property 5.1: Given a location vector Z used to define DC and NBC, the following are equivalent:

(a) Z is efficient,

(b) Each N_i is in at least one tight path in NBC,

(c) Z is the unique solution to DC,

(d) $z_i \varepsilon H[A_i(Z)]$ for $i = 1, \ldots, m$.

Proof: The equivalence of (b) and (c) is a direct consequence of Property 3.1 and the fact that Z is always a feasible solution to DC, while (c) clearly implies (a). To show (a) implies (b), suppose some N_i is not in a tight path. Property 3.3 then implies some entry in U = D(Z) can be reduced and the resultant distance constraints will still have a feasible solution, say Y. But then clearly $D(Y) \leq D(Z)$ and $D(Y) \neq D(Z)$, contradicting the fact that Z is efficient. Hence (a), (b), and (c) are equivalent. It can be seen that the proof will be complete if we show (b) implies (d) and (d) implies (c).

To show (b) implies (d), suppose N_i is in some tight path P. Let f_1 and f_2 be the nodes adjacent to N_i in P, so that $((f_1, N_i), (N_i, f_2))$ is a subpath of P. Since f_1 and f_2 are in the tight path P, by Property 3.2 the facilities represented by f_1 and f_2 are uniquely located. We may let y_1 and y_2 denote the unique locations of f_1 and f_2 respectively. Thus it is clear that y_1 and y_2 are elements of $A_i(Z)$. By Property 3.2, $z_i \in L(y_1, y_2)$, and by definition of the convex hull, $L(y_1, y_2) \subset H(A_i(Z))$. Thus it follows that $z_i \in H(A_i(Z))$. To show (d) implies (c), suppose $z_i \in H[A_i(Z)]$ and let f_1 and f_2 be nodes adjacent to N_i in NBC, where f_1 and f_2 represent facilities with unique locations y_1 and y_2, respectively, such that $z_i \in L(y_1, y_2) \subset T$. Thus $d(y_1, y_2) = d(y_1, z_i) + d(z_i, y_2)$. Now for any feasible solution X to DC we know $d(y_1, x_i) \leq d(y_1, z_i)$ and $d(y_2, x_i) \leq d(y_2, z_i)$. But then because f_1 and f_2 are uniquely located, Lemma 3.2 implies $x_i = z_i$, for i = 1, ..., m. Hence X = Z, so Z is the unique solution to DC, completing the proof.

As an example, suppose we again have three existing and three new facilities, and use the tree of Fig. 1. Suppose the entries of D(Z) are given by $d(z_1, a_1)$, $d(z_1, a_3)$, $d(z_2, a_1)$, $d(z_2, a_3)$, $d(z_3, a_1)$, $d(z_3, a_2)$ $d(z_1, z_2)$, $d(z_1, z_3)$, and $d(z_2, z_3)$. Suppose the location vector Z is as shown in Fig. 3. Fig. 3 and the above distances gives NBC as shown in Fig. 4. As the path (E_1, N_1, E_3) in NBC has length $6 = d(a_1, a_3)$, N_1 is on a tight path. Further, the path (E_1, N_2, E_3) also has length $6 = d(a_1, a_3)$, so N_2 is on a tight path. However, it can be verified that every path containing N_3 is slack, so the vector $Z = (z_1, z_2, z_3)$, is dominated. It is easily verified however that if z_3 is changed so that either $z_3 \varepsilon L(a_1, a_2)$ or $z_3 \varepsilon L(z_1, z_2)$, that the new location vector will be efficient, as N_3 is on at least one tight path, or, equivalently, $z_3 \varepsilon H(\{a_1, a_2, z_1, z_2\})$.

As a final application of the idea of efficiency, let us choose an efficient location vector problem and an efficient location vector Z for which $U = D(Z)$ will have $m(m-1)/2 + mn$ entries. Suppose we subtract any positive quantity ε from an arbitrary entry in U to obtain a vector U_ε, giving $U_\varepsilon \leq U$ and $U_\varepsilon \neq U$. The distance constraints $D(X) \leq U_\varepsilon$ must then be inconsistent, as else X would dominate Z. Now given the constraints $D(X) \leq U_\varepsilon$, if we do not know which entry in U has been reduced by ε to obtain U_ε, then any algorithm we apply to check if $D(X) \leq U_\varepsilon$ is consistent must examine <u>all</u> of the $m(m-1)/2 + mn$ constraints, since <u>any</u> <u>one</u> of the constraints can cause inconsistency. Hence any algorithm to determine if $D(X) \leq U_\varepsilon$ is consistent is at least of order $m^2 + mn$. Thus we have

<u>Property 5.2</u>: Any algorithm to determine whether or not the distance constraints of Section 2 are consistent is (for the worst case) at least of order $m^2 + mn$.

As is pointed out in [8], the "Sequential Location Procedure" presented in [8] to check the consistency of DC is of order $m^2 + mn$, and thus

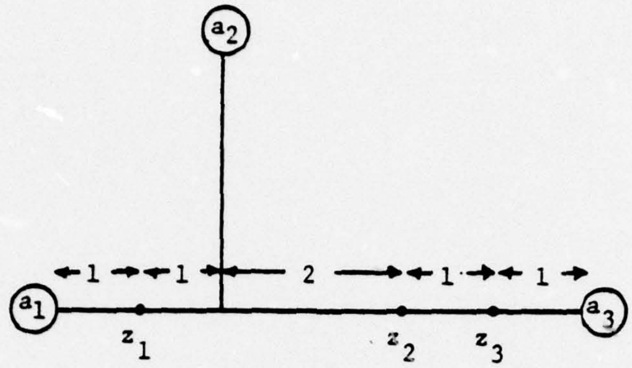

Figure 3. Given Locations for Example Multiobjective Problem

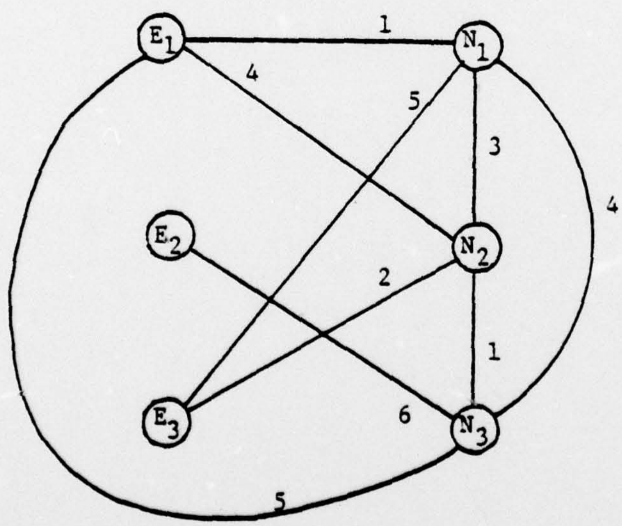

Figure 4. NBC for Example Multiobjective Problem

Property 5.2 implies it is in fact an algorithm of lowest order for checking consistency.

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